Cavity Method for Random Graphs

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Spin Glasses and Disorder

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Cavity method:

- iterative method on tree-like structures
- mean field type approach
- notion of neighborhood (graphs with finite connectivity)
- equivalent to Replica approach (RS, 1RSB, ...)

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Graphs:
1. Cayley tree
   - strongly inhomogeneous
   - frustration → boundary conditions
2. random graph
   - fixed connectivity
   - random connectivity
3. Bethe lattice
   - inside of a Cayley tree → ambiguous in presence of frustration
   - random lattice with fixed connectivity $c = k + 1$
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Basic operations:
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Basic operations:

**Iteration:** pick $k$ cavity spins and add a new cavity spin

\[ Z = \prod_{i=1}^{k} Z_i \quad \text{and} \quad Z_i = \sum_{\sigma_i} Z_i(\sigma_i) \]
Cavity method: \( q \) cavity spins have only \( k \) neighbors (connectivity \( c = k + 1 \))

Basic operations:

Iteration: pick \( k \) cavity spins and add a new cavity spin \( \rightarrow \delta N = 1, \delta q = -k + 1 \)

\[
Z_0 = \sum_{\{\sigma_i\}, \sigma_0} \prod_{i=1}^{k} Z_i(\sigma_i) e^{-\beta \mathcal{H}(\sigma_i, \sigma_0; J_i)}
\]

\[
-\beta \Delta F^{(\text{iter})} = \log \frac{Z_0}{\prod_{i=1}^{k} Z_i}
\]
Cavity method: *q cavity spins* have only *k* neighbors (connectivity *c = k + 1*)

**Basic operations:**

**Site addition:** pick *k + 1* cavity spins and add a spin

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Z = \prod_{i=1}^{k+1} Z_i \quad \quad Z_i = \sum_{\sigma_i} Z_i(\sigma_i)
\]
Cavity method: \( q \) cavity spins have only \( k \) neighbors (connectivity \( c = k + 1 \))

Basic operations:

Site addition: pick \( k + 1 \) cavity spins and add a spin \( \rightarrow \delta N = 1, \delta q = -k - 1 \)

\[
Z_0 = \sum_{\{\sigma_i\}, \sigma_0} \prod_{i=1}^{k+1} Z_i(\sigma_i) e^{-\beta \mathcal{H}(\sigma_i, \sigma_0; J_i)}
\]

\[
-\beta \Delta F^{(\text{site})} = \log \frac{Z_0}{\prod_{i=1}^{k+1} Z_i}
\]
**Cavity method:** $q$ cavity spins have only $k$ neighbors (connectivity $c = k + 1$)

**Basic operations:**

**Link addition:** pick 2 cavity spins and add a link

$$Z = Z_i Z_j$$
$$Z_i = \sum_{\sigma_i} Z_i(\sigma_i)$$
Cavity method: \( q \) cavity spins have only \( k \) neighbors (connectivity \( c = k + 1 \))

Basic operations:

Link addition: pick 2 cavity spins and add a link \( \rightarrow \delta N = 0, \delta q = -2 \)

\[
Z_0 = \sum_{\sigma_i, \sigma_j} Z_i(\sigma_i) Z_j(\sigma_j) e^{-\beta H(\sigma_i, \sigma_j; J_{ij})}
\]

\[
-\beta \Delta F^{\text{link}} = \log \frac{Z_0}{Z_i Z_j}
\]
**Free energy:** in the large $N$ limit, $F_N$ is linear in $N$

$$f(\beta) = \lim_{N \to \infty} \frac{F_N(\beta)}{N} = \frac{F_{N+2} - F_N}{2} = \Delta F_{\text{site}} - \frac{k + 1}{2} \Delta F_{\text{link}}$$
**Free energy:** in the large $N$ limit, $F_N$ is linear in $N$

$$f(\beta) = \lim_{N \to \infty} \frac{F_N(\beta)}{N} = \frac{F_{N+2} - F_N}{2} = \Delta F^{(\text{site})} - \frac{k + 1}{2} \Delta F^{(\text{link})}$$

- $f(\beta) = \frac{1}{N} \left( \sum_i \Delta F_i^{(\text{site})} - \sum_{(ij)} \Delta F_{(ij)}^{(\text{link})} \right)$
- $f(\beta) = \frac{k + 1}{2} \Delta F^{(\text{2sites})} - k \Delta F^{(\text{site})}$
- ...
Models:

1. **Ising spins** $\sigma_i = \pm 1$ with random couplings

   \[ \mathcal{H} = - \sum_{(ij)} J_{ij} \sigma_i \sigma_j \]

   **Cavity fields:**
   \[ Z_i(\sigma_i) = \exp(\beta \sigma_i h_i) \]

   Belief propagation:
   \[
   \sum_{\{\sigma_i\}} \exp \left( \beta \sigma_0 \sum_{i=1}^{k} J_i \sigma_i + \beta \sum_{i=1}^{k} h_i \sigma_i \right) = \left[ \prod_{i=1}^{k} c(J_i, h_i) \right] \exp \left( \beta \sum_{i=1}^{k} u(J_i, h_i) \sigma_0 \right) \]

2. **Potts spin model** $s_i = 1, \ldots, q$

   with antiferromagnetic couplings

   \[ \mathcal{H} = \sum_{(ij)} \delta(s_i, s_j) \]

   **Cavity fields (probabilities):**
   \[ \psi_{s_i}^j = \frac{Z_i(s_i)}{Z_i} \]

   Belief propagation:
   \[
   \psi_{s_0}^0 = \frac{1}{Z^{(\text{iter})}} \prod_{i=1}^{k} \left[ 1 - (1 - e^{-\beta}) \psi_{s_0}^i \right] \]
Models:

1. Ising spins $\sigma_i = \pm 1$ with random couplings

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2. Potts spin model $s_i = 1, \ldots, q$

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Cavity fields (probabilities): $\psi^i_{s_i} = \frac{Z_i(s_i)}{Z_i}$

Belief propagation:

$$\psi^0_{s_0} = \frac{1}{Z^{\text{iter}}} \prod_{i=1}^{k} \left[ 1 - (1 - e^{-\beta}) \psi^i_{s_0} \right]$$
Replica Symmetric solution

Ensemble of random cavity graphs $\rightarrow$ cavity fields: i.i.d. random variables with distribution $\mathcal{P}(\psi)$

$$
\mathcal{P}(\psi) = \int \left[ \prod_{i=1}^{k} d\psi^i \mathcal{P}(\psi^i) \right] \delta \left( \psi - \mathcal{F}(\{\psi^i\}) \right)
$$

$\mathcal{P}(\psi) \rightarrow$ fluctuations from site to site
$\rightarrow$ quenched averages (over the ensemble of random graphs) of all thermodynamical quantities (free energy, overlaps,...)

**RS instability:**

- high-$T$, paramagnetic phase: single pure state $\rightarrow$ RS
- $T_{\text{local}}$: divergence of spin-glass susceptibility $\chi_{SG} = \frac{1}{N} \sum_{i,j} \langle s_is_j \rangle_c$
  \[
  T_{\text{local}} = -\frac{1}{\log \left( 1 - \frac{q}{\sqrt{k+1}} \right)}
  \]
  - spin glass solution appears before $T_{\text{local}} \rightarrow T_d$: dynamical phase transition
One step Replica Symmetry Breaking

RS approach neglects the possibility of the existence of many pure states
we now take into account many pure states labeled by $\alpha$

1RSB - hypothesis:

- state $\alpha$: fixed point of the belief propagation $\{\psi\}$
  (cavity spins uncorrelated within one pure state)

- on a given site $i$, $\psi^i_\alpha$:
  i.i.d. with $P_i(\psi) \rightarrow$ fluctuates from site to site $\rightarrow \mathcal{P}[P(\psi)]$

- free energies of the states on one branch:
  i.i.d. random variable $\rho(F) = \exp [\beta m (F - F_R)]$

- pure states have statistical weight $W^\alpha = \frac{\exp(-\beta F^\alpha)}{\sum_\gamma \exp(-\beta F^\gamma)}$

  $\rightarrow m < 1$

  $\rightarrow$ probability measure over state: $\mu_\alpha = \frac{(Z^\alpha)^m}{Z_1} = \frac{1}{Z_1} e^{-\beta m F^\alpha}$

Self-consistency under iteration
Iterative equations:

\[ \mathcal{P}[P(\psi)] = \int \left[ \prod_{i=1}^{k} dP_i(\psi^i) \mathcal{P}[P_i(\psi^i)] \right] \delta \left( P(\psi) - \mathcal{F}_2(\{P_i(\psi^i)\}) \right) \]

where

\[ \mathcal{F}_2(\{P_i(\psi^i)\}) = \frac{1}{Z_1^{(\text{iter})}} \int \left[ \prod_{i=1}^{k} P_i(\psi^i)d\psi^i \right] \delta \left( \psi - \mathcal{F}(\{\psi^i\}) \right) e^{-\beta m \Delta F^{(\text{iter})}(\{\psi^i\})} \]

Thermodynamical quantities:

ex: site contribution to free energy

\[ \Delta F^{(\text{site})} = \]

Numerical method: population dynamics
Iterative equations:

$$\mathcal{P}[P(\psi)] = \int \left[ \prod_{i=1}^{k} dP_i(\psi^i) \mathcal{P}[P_i(\psi^i)] \right] \delta \left( P(\psi) - \mathcal{F}_2(\{P_i(\psi^i)\}) \right)$$

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Thermodynamical quantities:

ex: site contribution to free energy

$$\Delta F^{(\text{site})} = \int \left[ \prod_{i=1}^{k+1} P_i(\psi^i) d\psi^i \right] e^{-\beta m \Delta F^{(\text{site})}}$$

Numerical method: population dynamics
Iterative equations:

\[
P[P(\psi)] = \int \left[ \prod_{i=1}^{k} dP_i(\psi^i) P[P_i(\psi^i)] \right] \delta \left( P(\psi) - \mathcal{F}_2(\{P_i(\psi^i)\}) \right)
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where

\[
\mathcal{F}_2(\{P_i(\psi^i)\}) = \frac{1}{Z_1^{(\text{iter})}} \int \left[ \prod_{i=1}^{k} P_i(\psi^i) d\psi^i \right] \delta \left( \psi - \mathcal{F}(\{\psi^i\}) \right) e^{-\beta m \Delta F^{(\text{iter})}(\{\psi^i\})}
\]

Thermodynamical quantities:

ex: site contribution to free energy

\[
\Delta F^{(\text{site})} = \int \left[ \prod_{i=1}^{k+1} dP_i P[P_i] \right] \frac{\int \left[ \prod_{i=1}^{k+1} P_i(\psi^i) d\psi^i \right] \Delta F^{(\text{site})} e^{-\beta m \Delta F^{(\text{site})}}}{\int \left[ \prod_{i=1}^{k+1} P_i(\psi^i) d\psi^i \right] e^{-\beta m \Delta F^{(\text{site})}}}
\]

Numerical method: population dynamics
Complexity

probability measure over state: \[ \mu_\alpha = \left( \frac{Z_\alpha}{Z_1} \right)^m = \frac{1}{Z_1} e^{-\beta m F^\alpha} \]

**Replicated free energy:**

\[ \Phi(\beta, m) \equiv -\frac{1}{\beta m N} \ln(Z_1) = \frac{1}{N} \left( \Delta \Phi^{(\text{site})} - \frac{k + 1}{2} \Delta \Phi^{(\text{link})} \right) \]

where:
\[ e^{-\beta m \Delta \Phi^{(\text{site})}} = \int_{\text{POP}} e^{-\beta m \Delta F^{(\text{site})}} \quad e^{-\beta m \Delta \Phi^{(\text{link})}} = \int_{\text{POP}} e^{-\beta m \Delta F^{(\text{link})}} \]

**Configurational entropy:** \[ \mathcal{N}(f) = \exp[N\Sigma(f)] \]

**Legendre transform:**
\[ Z_1 = e^{-\beta m N \Phi(\beta, m)} = \sum_\alpha e^{-\beta m N f^\alpha} = \int df e^{-N[\beta mf - \Sigma(f)]} \]

**saddle point:**
\[ -\beta m \Phi(\beta, m) = -\beta mf(\beta) + \Sigma(f) \]
Zero temperature limit:

1. **Energetic zero-\( T \) limit:** \( \beta \to \infty, \ y = \beta m \ \text{const.} \)

\[
-y \Phi_\epsilon(y) = -y\epsilon + \Sigma(\epsilon)
\]

\( \Sigma(\epsilon) \): number of local ground states (LGS) of energy \( \epsilon \)

- Ising model: \( \Sigma(\epsilon) = 0 \) \( \to \) global ground state
- Potts model: \( \Sigma(0) = 0 \) \( \to \) coloring threshold

2. **Entropic zero-\( T \) limit:** \( \epsilon = 0, \ \Phi_s(m) = -\beta m \Phi(\beta, m)|_{\beta \to \infty} \)

\[
\Phi_s(m) = ms + \Sigma(s)
\]

\( \Sigma(s) \): number of clusters of size \( s \)

\( \to \) more information on the phase space of zero-energy state (coloring)
Potts model: some results

**m-T diagram** [Zdeborova, Krzakala, EPL (2008)]

1. **shaded region:** non trivial (non RS) solution exists $\rightarrow$ at $m = 1$, $T < T_d$

2. **full free energy:** $-\beta f_{1RSB} = \max_{f: \Sigma(f) \geq 0} \left[ -\beta f + \Sigma(f) \right] \quad \rightarrow \quad m^*(T)$
Potts model: some results

*m*-*T* diagram [Zdeborova, Krzakala, EPL (2008)]

1. **$T_K < T < T_d$:** $\Sigma(m = 1) > 0 \Rightarrow m^* = 1 \rightarrow f_{1RSB} = f_{RS}$

2. $T < T_K$: $\Sigma(m = 1) < 0 \Rightarrow f^* : \Sigma(f^*) = 0 \rightarrow f_{1RSB} > f_{RS}$

3. $T_K < T < T_d$: $\Sigma(m = 1) > 0 \Rightarrow m^* = 1 \rightarrow f_{1RSB} = f_{RS}$

4. if $m^*(T = 0) > 0$: the graph is $q$-colorable

5. if $m^*(T = 0) = 0$: the graph is not $q$-colorable
Potts model: some results

\textit{m-T diagram} [Zdeborova, Krzakala, EPL (2008)]

5 Gardner instability:

- small changes in site dependent distribution $P(\psi)$: dashed line
- small changes in $\psi$: dark gray region
  $\xrightarrow{}$ more steps of RSB are necessary
Potts model: some results

\(c-T\) diagram [Zdeborova, Krzakala, EPL (2008)]

- \(q = 3\): \(T_{\text{local}} = T_d = T_K\)
- \(q \geq 4\): \(T_d, T_K \downarrow T_{\text{local}}\)
- bold red line at \(T = 0\): uncolorable phase \(c > c_s\)
- colorable phase and threshold are stable
Zero temperature limit:

entropic zero- $T$ limit $\rightarrow$ same analysis as for finite $T$

\begin{itemize}
  \item $c < c_d$: no nontrivial solution at $m = 1$
  \item $c_d \leq c < c_c$: nontrivial solution $\Sigma(s) > 0$ with $m^* = 1$
  \item $c_c \leq c < c_s$: $\Sigma(m = 1) < 0$, then $m^* : \Sigma(m^*) = 0$
  \item $c_s \leq c$: $\Sigma(s) < 0$, uncolorable phase
\end{itemize}

Figure: Zdeborova, Krzakala, PRE (2007)

c_d = 18: clustering (dynamical) transition

c_c = 19: condensation transition (disc.)

c_s = 20: COL/UNCOL transition
- $c < c_d$: no nontrivial solution at $m = 1$
- $c_d < c < c_c$: nontrivial solution $\Sigma(s) > 0$ with $m^* = 1$
- $c_c < c < c_s$: $\Sigma(m = 1) < 0$, then $m^* : \Sigma(m^*) = 0$
- $c > c_s$: $\Sigma(s) < 0$, uncolorable phase
1RSB: Self-consistency of the hypothesis

\((\psi_0^\alpha, \Delta F^\alpha)\): i.i.d taken from \(P_0(\psi^0, \Delta F)\)

\[
P_0(\psi^0, \Delta F) = \int \left[ \prod_{i=1}^k P_i(\psi^i)d\psi^i \right] \delta \left( \psi^0 - \mathcal{F}(\{\psi^i\}) \right) \delta \left( \Delta F - \Delta F^{(\text{iter})}(\{\psi^i\}) \right)
\]

Joint distribution \(R_0(\psi^0, F')\):

\[
R_0(\psi^0, F') \propto \int d(\Delta F)dF \ P_0(\psi^0, \Delta F) \ e^{\beta m(F - F_R)} \ \delta \left( F' - F - \Delta F \right)
\]

\[
\propto e^{\beta m(F' - F_R)} P_0(\psi^0)
\]

where

\[
P_0(\psi^0) \propto \int d(\Delta F) P_0(\psi^0, \Delta F) \ e^{-\beta m \Delta F}
\]

\[
\propto \int \left[ \prod_{i=1}^k P_i(\psi^i)d\psi^i \right] \delta \left( \psi^0 - \mathcal{F}(\{\psi^i\}) \right) \ e^{-\beta m \Delta F^{(\text{iter})}(\{\psi^i\})}
\]
Numerical method: population dynamics

**Population dynamics:** \( \mathcal{M} \) fields to represent \( P_i(\psi) \)

\[ \mathcal{N} \] populations of \( \mathcal{M} \) fields to represent \( \mathcal{P}[P_i(\psi)] \)

Iteration

1. choose \( k \) sites at random
2. for each of the \( \mathcal{M} \) states, compute \( \psi_\alpha^0 \) and \( \Delta F_\alpha^{\text{iter}} \), \( \alpha = 1, \ldots, \mathcal{M} \)
3. take into account the weight: \( \exp \left( -\beta m \Delta F_\alpha^{\text{iter}} \right) \rightarrow \psi_\alpha \)
4. substitute sequentially population on site \( i \) with the new \( \psi^i \)

\[ \rightarrow \text{Markov chain} \] on the space of the \( \mathcal{N} \) sites

after a transient \( \rightarrow \psi_\alpha \) distributed according to \( P(\psi) \)

**Stationary state** \( \rightarrow \) compute all thermodynamical quantities

uniformly sampling over the population
Zero-\(T\) limit: example

**Ising model** (\(J_{ij} = \pm J\)): energetic zero-\(T\) limit

- non-factorized solution
- right branch: physical
- left branch: unphysical

**Figure:** Mezard, Parisi, EPJ (2003)
References


L. Zdeborova, M. Krzakala, “Potts glass on random graphs”, EPL 81, 57005 (2008)